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Spontaneous emission of bound photons from relativistic free electrons

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If the states of photons are bound, another type of spontaneous emission becomes possible for a relativistic electron due to the symmetry breaking in space. We obtain the radiation probability for a relativistic electron passing through a box-shaped cavity. The radiation spectrum is discrete in which line positions are determined by the boundary condition of the cavity.

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It is well known that a free electron in vacuum cannot emit a photon spontaneously because energy and momentum are not conserved simultaneously. However, if the electron is in a binding state at least in one direction, spontaneous emission becomes possible. Free electron laser, channeling radiation, and coherent bremsstrahlung may be classified into this kind of spontaneous emission [1]. In these processes the momentum is not conserved in the direction where the radiating electron is bound.

In this paper, we show that yet another type of spontaneous emission becomes possible when the emitted photons are in binding states, such as those in a cavity or waveguide. For example, if an electron passes through a cavity as shown in Fig. 1, it may emit photons spontaneously because photons do not have definite momentum and hence the restriction coming from momentum conservation does not exist. In this case, the electron is virtually free, unlike the "free electron" laser in which the motion of electrons is in fact sinusoidal or spiral.

Let us derive the probability of the "cavity radiation." As usual, the interaction between the photon field and a relativistic electron is given by

$$H'(t) = -e\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r},t), \tag{1}$$

where $\boldsymbol{\alpha}$ is the ordinary Dirac matrix and $\mathbf{A}(\mathbf{r})$ is the photon field,

$$\mathbf{A}(\mathbf{r},t) = \sum_{s,\sigma} [\mathbf{u}_{s,\sigma}(\mathbf{r})a_{s,\sigma}(t) + \mathbf{u}^*_{s,\sigma}(\mathbf{r})a^\dagger_{s,\sigma}(t)], \tag{2}$$

where $a_{s,\sigma}(t)$ is the annihilation operator of the photon in the mode s and polarization σ. $\mathbf{A}(\mathbf{r},t)$ is normalized as

$$\frac{1}{8\pi}\int_V d^3\mathbf{r}(|\mathbf{E}|^2 + |\mathbf{H}|^2) = \sum_{s,\sigma} \hbar\omega_{s,\sigma}\left(a^\dagger_{s,\sigma}a_{s,\sigma} + \frac{1}{2}\right), \tag{3}$$

where $\mathbf{E} = -\partial\mathbf{A}/c\partial t$ and $\mathbf{H} = \mathrm{rot}\mathbf{A}$ are the electromagnetic field in the cavity, V the volume of the cavity where photons are bound, $\hbar\omega_{s,\sigma}$ the energy of the photon in the mode s and the polarization direction σ, and $a_{s,\sigma}(t) = a_{s,\sigma}\exp(-i\omega_{s,\sigma}t)$.

In this paper, for the purpose of clarifying the idea of the radiation process, we consider the radiation field in a box as shown in Fig. 1. This is one of the simplest cases where the photons are in the bound states. Also, we assume that the absorption and transmission rates of photons at the boundary are small. This assumption is valid for an extracting mirror of which transmittivity is about a few percent. In this case, $\mathbf{u}(\mathbf{r})$ becomes the well-known wave function in a box-shaped resonant cavity,

$$\begin{aligned}\mathbf{u}_{s,\sigma}(\mathbf{r}) = \sqrt{\frac{16\pi\hbar c^2}{XYZ\omega}}[&\mathbf{e}^x_{s,\sigma}\cos(k_l x)\sin(k_m y)\sin(k_n z)\\ &+ \mathbf{e}^y_{s,\sigma}\sin(k_l x)\cos(k_m y)\sin(k_n z)\\ &+ \mathbf{e}^z_{s,\sigma}\sin(k_l x)\sin(k_m y)\cos(k_n z)][\theta(x) - \theta(x-X)]\\ &\times[\theta(y) - \theta(y-Y)][\theta(z) - \theta(z-Z)],\end{aligned} \tag{4}$$

where $\theta(x)$ is the step function, XYZ is the volume of the cavity, and $s = (l,m,n)$ is the "quantum number" indicating the state of a photon in the cavity. $\mathbf{e}^i_{s,\sigma}$ $(i = x,y,z)$ represents the polarization vector in the i direction. $k_l = \pi l/X$, $k_m = \pi m/Y$, and $k_n = \pi n/Z$ satisfy $(\omega_{lmn}/c)^2 = k_l^2 + k_m^2 + k_n^2$. It is straightforward to extend our theory to any type of cavities or waveguides. The only thing one has to do is to calculate $\mathbf{A}(\mathbf{r})$ for suitable boundary conditions.

The number of emitted photons with energy $\hbar\omega_{lmn}$ may be calculated by the golden rule:

$$\begin{aligned}N_{lmn} = \frac{2\pi Z}{\hbar v}\sum_{\mathbf{p}'}\sum_{\sigma}\frac{1}{2}\sum_{\nu,\nu'}&|\langle 1^\sigma_{lmn},\mathbf{p}'\nu'|(-e)\boldsymbol{\alpha}\cdot\mathbf{A}|0,\mathbf{p}\nu\rangle|^2\\ &\times\delta(E_{\mathbf{p}} - E_{\mathbf{p}'} - \hbar\omega_{lmn}),\end{aligned} \tag{5}$$

where $E_{\mathbf{p}}$ and $\mathbf{p}$ are the energy and momentum of the electron and $|n^\sigma_{lmn}\rangle$ is the eigenvector of the Hamiltonian for the

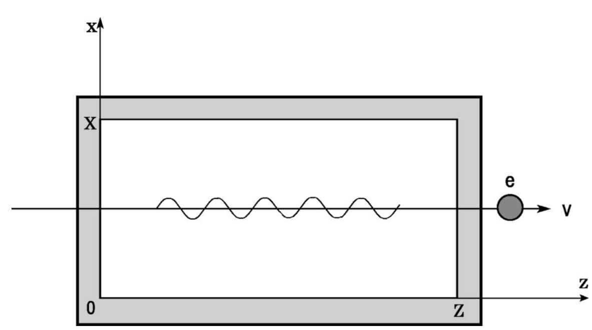

FIG. 1. A relativistic electron passing through a box-shaped cavity composed of mirrors. Due to the symmetry breaking in space, the free electron may emit a bound photon spontaneously.

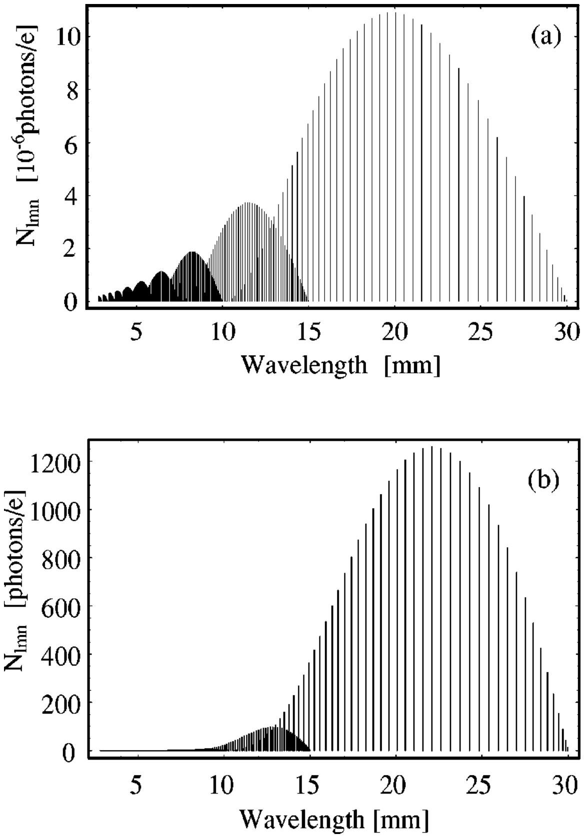

FIG. 2. Number of photons N_{lmn} in the box-shaped resonant cavity irradiated with a 30-MeV electron as a function of the photon wavelength. Only the modes $l=0$ are shown. (a) represents the photon spectrum emitted by a single electron whereas (b) shows $N_{lmn}^{(bunch)}/n_e$, i.e., the number of photons per electron in a bunch, $d=5$ mm and $n_e=10^7$ (see the text).

radiation field representing that the number of photons in the state lmn,σ is *n*. The wave function for the electron is given by $\langle \mathbf{r}|\mathbf{p}\nu\rangle = u_\nu(\mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{r}/\hbar)$, where the spinor $u_\nu(\mathbf{p})$ is normalized in a box so as to give $\langle \mathbf{p}'\nu'|\mathbf{p}\nu\rangle = \delta_{\mathbf{p}',\mathbf{p}}\delta_{\nu,\nu'}$. We have assumed that the normalization box of the electron is the same as that of photons in the xy direction, while it is much longer in the *z* direction. We have also assumed that the electron is moving along the *z* axis (see Fig. 1).

Substituting Eqs. (2)–(4) into Eq. (5), we obtain after some calculations

$$N_{lmn} = \frac{8\pi e^2 v^2}{XYZ\hbar\omega_{lmn}} \frac{\omega_{lmn}^2 - k_n^2 c^2}{(\omega_{lmn}^2 - k_n^2 v^2)^2} \times \left[1 - (-1)^n \cos\left(\frac{\omega_{lmn}}{v} Z\right)\right]. \tag{6}$$

In deriving Eq. (6), we have assumed that $\hbar\omega_{lmn} \ll E_{\mathbf{p}}$. A typical radiation spectrum calculated from Eq. (6) is given in Fig. 2. We have chosen the size of the cavity as $Z=15, X=210, Y=310$ mm so that ω_{lmn} is not degenerate in the shown range of the wavelength.

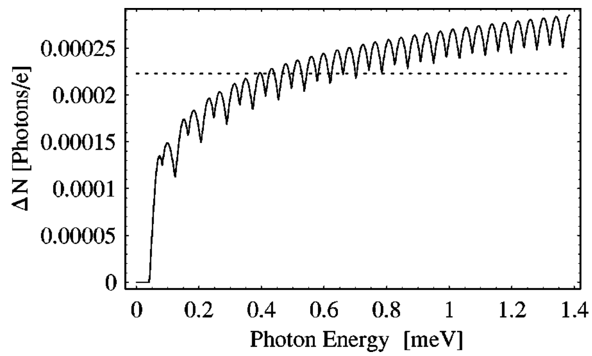

FIG. 3. Number of photons $\Delta N = (dN/d\omega)_{cav}\Delta\omega$ as a function of photon energy in the one-dimensional cavity limit. The dotted line shows the number of photons for transition radiation. For $\Delta\omega$ we have taken $\Delta\omega/\omega = 10^{-2}$. Other parameters needed for the numerical calculation are the same as Fig. 2.

As seen from Eq. (6), if the particle is sufficiently relativistic (i.e., satisfying $v \approx c$), the number of emitted photons does not depend on the energy of the incident electron.

Though radiation spectra depend on the shape of the cavity, Eq. (6) approaches to a certain continuous spectrum limit if we take $X,Y \to \infty$. It is convenient to compare the intensity of the ''one-dimensional cavity limit'' to other radiation processes such as transition radiation. In this limit, we have $N_{lmn} \approx N_n(\mathbf{k}_\perp)d^2\mathbf{k}_\perp$ because the frequency of photons may be regarded as a continuous variable. Then, by using the relation $c^2 d^2\mathbf{k}_\perp = \omega d\omega d\phi$ and integrating Eq. (6) over the azimuth angle ϕ, we obtain [6]

$$\frac{dN_n(\omega)}{d\omega} = \frac{4e^2v^2}{\hbar c^2 Z} \frac{\omega^2 - k_n^2 c^2}{(\omega^2 - k_n^2 v^2)^2}\left[1 - (-1)^n \cos\left(\frac{\omega}{v} Z\right)\right], \tag{7}$$

where $\omega = c\sqrt{\mathbf{k}_\perp^2 + k_n^2}$.

In experiments, the total number of photons

$$\left(\frac{dN(\omega)}{d\omega}\right)_{\text{cav}} = \sum_n^\infty \frac{dN_n(\omega)}{d\omega} \tag{8}$$

will be actually observed. In Fig. 3, we show $\Delta N = (dN/d\omega)_{cav}\Delta\omega$ taking the summation up to $n=120$. This number is enough to obtain a converged value for Fig. 3. For comparison, the transition radiation probability [2]

$$\left(\frac{dN(\omega)}{d\omega}\right)_{\text{TR}} = \frac{e^2}{\hbar c \omega}[\ln(4\gamma^2) - 1] \tag{9}$$

is also shown as the dotted line in Fig. 3. For wavelengths satisfying the condition $\lambda \ll Z$, the total number of photons of Eq. (8) approaches to the same order as that of transition radiation, Eq. (9). This fact may be explained by taking the continuous frequency limit in the following way. At $n \gg 1$, the terms with the factor $(-1)^n$ in Eq. (7) oscillate rapidly and hence after the summation they are canceled [7]. Then, by changing the summation to the integration and assuming that $k_1 = \pi/Z \ll \omega/c$, we obtain

$$\left(\frac{dN(\omega)}{d\omega}\right)_{\rm cav} \approx \frac{4e^2v^2}{\hbar c^2 Z}\left(\frac{Z}{\pi}\right)\int_{k_1}^{\omega/c} dk \frac{1}{\omega^2 - k^2 v^2} \approx \frac{2e^2}{\hbar c \omega} \ln \gamma^2. \tag{10}$$

This value is about twice as large as that of Eq. (9). With this correspondence, one may think that the cavity radiation is a kind of transition radiation. This is valid only in the sense that transition radiation, parametric x-ray radiation, the Smith-Purcell effect, and diffraction radiation are all the same, in that their origin is the space inhomogeneity of the dielectric response function [1,2]. Of course, we usually distinguish, for example, parametric x-ray radiation from transition radiation because the former is based on the lattice periodicity while the latter is based on the discontinuity at the surface. In this sense, the bound photon emission considered here is different from transition radiation. One of the most significant differences is that the spectra of cavity radiation depend on the sidewalls of the cavity. In the one-dimensional cavity limit, the cavity radiation reduces to transition radiation because the effect of the sidewalls disappears.

For a possible application of the cavity radiation, it is interesting to consider a compact source of strong far-infrared light. In this case, there appears a coherent effect when $\lambda \gtrsim d$, where d represents the size of the bunch of a pulsed electron beam. In this case, radiation probability is enhanced as [3]

$$N_{lmn}^{(\rm bunch)} = n_e N_{lmn}(1 + n_e|\tilde{\rho}(k_n)|^2) \approx n_e^2 N_{lmn}\left|\int \rho(z) e^{ik_n z} dz\right|^2, \tag{11}$$

where n_e is the total number of electrons in the bunch and $\rho(z)$ is the density distribution along the beam direction. If $\lambda \gg d$, which is satisfied at centimeter wavelengths, then we have $\tilde{\rho}(k_n) \sim \tilde{\rho}(0) = 1$, where $\tilde{\rho}(k)$ is the Fourier transform of $\rho(z)$. In this case $N_{lmn}^{(\rm bunch)}$ has a perfect coherence and it becomes n_e times stronger than the incoherent case. Since n_e is a huge number, radiation is enhanced by many orders of magnitude. An example of this bunch effect is shown in Fig. 2(b). The number of photons, per one electron in the bunch is enhanced n_e times at wavelengths satisfying $\lambda \gg d$. For such a large number of photons the induced emission may arise [4] in the cavity with the use of the pulsed beam. In Ref. [5] related phenomena were reported, although the photons were emitted as synchrotron radiation. It is interesting to note that in Ref. [5], the authors mentioned that even without the magnetic field they observed line spectra. The authors attributed this radiation to transition radiation. However, since the formation length of transition radiation is of the order of 10 m at the millimeter wavelengths, the radiation from each mirror cannot be considered independently. One must consider the radiation process coherently as discussed above.

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[6] The factor $[1-(-1)^n\cos(\omega Z/v)]$ in Eq. (7) is related to the well-known interference factor $[1-\cos(Z/l_f)]$ [2,3] in the following way: since $k_z Z - n\pi = 0$, we may have $\omega Z/v = \omega Z/v - k_z Z + n\pi = Z/l_f + n\pi$, where l_f is the formation length [1,2]. Then the factor becomes $[1-(-1)^n\cos(\omega Z/v)] = [1-(-1)^n\cos(Z/l_f + n\pi)] = [1-\cos(Z/l_f)]$.

[7] More precisely, in connection with Ref. [6], we should write the condition that the oscillating factor may be neglected as $Z \gg l_f(\omega)$. This condition is satisfied in the high-frequency limit.